

## The hitting distribution of line segments for two dimensional random walks

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**Abstract** Asymptotic estimates of the hitting distribution on a long segment on the real axis for two dimensional random walks on  $\mathbf{Z}^2$  of zero mean and finite variance are obtained.

## 1 Introduction and Results

Let  $S_n$  be a two dimensional random walk of i.i.d. increments on the square lattice  $\mathbf{Z}^2$ , which we suppose to be embedded in the complex plane  $\mathbf{C}$ . Let  $n$  be a positive integer and denote by  $H_z^{I(n)}(s)$  the probability that the first visit (after time 0) to the interval  $\{-n+1, \dots, n-1\}$  of the random walk  $S$  starting at  $z$  takes place at  $s$ . For the later use it is convenient to define  $n_*$  and  $I(n)$  by

$$I(n) = (-n_*, n_*) = \{u \in \mathbf{R} : |u| < n_*\}, \quad n_* = n - 1/2.$$

Then  $H_z^{I(n)}(s)$ ,  $s \in I(n)$ , is written as

$$H_z^{I(n)}(s) = P_z[\exists j \geq 1, S_j = s \text{ and } S_k \notin I(n) \text{ for } 1 \leq k < j],$$

where  $P_z$  stands for the probability of the walk starting at  $z \in \mathbf{Z} + i\mathbf{Z}$ . The corresponding distribution for Brownian motion is known explicitly. Let  $h_x^{I(n)}$  denote the Brownian analogue of  $H_z^{I(n)}$ , namely the density of hitting distribution of the interval  $I(n)$  for the two dimensional standard Brownian motion starting at  $x \in \mathbf{R} \setminus [-n_*, n_*]$ . Then,

$$h_x^{I(n)}(s) = \frac{\sqrt{x^2 - n_*^2}}{\pi|x - s|} \cdot \frac{1}{\sqrt{n_*^2 - s^2}} \quad (s \in I(n)) \quad (1)$$

(see Appendix).

For the symmetric simple random walk H. Kesten has obtained the upper bound  $\lim_{z \rightarrow \infty} H_z^{I(n)}(s) \leq C[n(n-s)]^{-1/2}$  ( $0 \leq s < n$ ) in [3] and applied it to a study of the DLA model in [4] (cf. also [5]; a unified exposition is found in [7]). This bound is extended and refined in the present paper. For a rectangle with a side on the real axis Lawler and Limic [8] give an explicit expression for the hitting distribution of its boundary for simple random walk started inside it and, by taking limits, derive from it the corresponding ones for a half-infinite strip and a quadrant.

Throughout this paper we suppose that the walk  $S_n$  is irreducible,  $E_0[S_1] = 0$  and

$$E_0[|S_1|^{2+\delta}] < \infty \text{ either for } \delta = 0 \text{ or for some } \delta > 1/2. \quad (2)$$

**Theorem 1** *Let  $\delta > 1/2$  in (2). Then uniformly for integers  $s \in I(n)$  and  $x, |x| \geq n$ , as  $n \rightarrow \infty$*

$$H_x^{I(n)}(s) = h_x^{I(n)}(s) \left[ 1 + O\left( \frac{1}{\sqrt{(|x| - n_*) \wedge (n - |s|)}} \right) \right].$$

Theorem 1 does not determine the asymptotic form of  $H^{I(n)}$  when either  $|x| - n_*$  or  $n - |s|$  remains bounded. The estimate of the following theorem improves on this respect in the case  $\delta = 0$ . (See Section 4 (Theorems 9, 10) for the case  $\delta > 1/2$ .) In [11] we have introduced a pair of functions  $\mu(y)$  and  $\nu(y)$ ,  $y \in \mathbf{Z}$  that are (strictly) increasing and satisfy that

$$\mu(-y)\sqrt{y} \rightarrow 1 \quad \text{and} \quad \mu(y)/\sqrt{y} \rightarrow 2/\sigma^2 \quad \text{as} \quad y \rightarrow \infty$$

and the same properties for  $\nu$  in place of  $\mu$ . Here  $\sigma^2$  is the square root of the determinant of the covariance matrix of  $S_1$  under  $P_0$ .

**Theorem 2** (i) *Uniformly for  $0 \leq s < n$  and  $x \geq n$ , as  $n \rightarrow \infty$  and  $x - s \rightarrow \infty$*

$$H_x^{I(n)}(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\nu(x - n)\mu(-n + s)}{x - s} \cdot \sqrt{\frac{x + n}{n + s}} (1 + o(1)).$$

(ii) *Uniformly for  $-n < s \leq 0$  and  $x \geq n$ , as  $n \rightarrow \infty$*

$$H_x^{I(n)}(s) = \frac{\sigma^2}{2\pi} \cdot \frac{\nu(x - n)\nu(-n - s)}{x - s} \cdot \sqrt{\frac{x + n}{n - s}} (1 + o(1)).$$

**Corollary 3** *If  $H_\infty^{I(n)} = \lim_{x \rightarrow \infty} H_x^{I(n)}$ , then uniformly for integers  $s \in I(n)$ , as  $n \rightarrow \infty$*

$$H_\infty^{I(n)}(s) = \pi^{-1} \mu(-n + s) \nu(-n - s) (1 + o(1)).$$

**Corollary 4** *Uniformly for integers  $n, s \in I(n)$  and  $x \in \mathbf{Z} \setminus I(n)$ ,  $H_x^{I(n)}(s) \asymp h_x^{I(n)}(s)$ , namely there exists a positive constant  $C$  independent of  $n, s$  and  $x$  such that*

$$C^{-1} h_x^{I(n)}(s) \leq H_x^{I(n)}(s) \leq C h_x^{I(n)}(s).$$

Denote by  $H_z^+(s)$  the probability that the first visit (after time 0) to the positive real axis of the walk starting at  $z \in \mathbf{C}$  takes place at  $s \in \{1, 2, 3, \dots\}$ :

$$H_z^+(s) = P_z[\exists n \geq 1, S_n = s \text{ and } S_k \in \{0, -1, -2, \dots\} \text{ for } 1 \leq k < n],$$

Similarly let  $H_z^-(s)$  denote the distribution of the first visiting sites (after time 0) of the set  $\{-1, -2, -3, \dots\}$ . The proofs of Theorems 1 and 2 are based on the results on  $H_x^\pm(s)$  obtained in [11] (as partly displayed in (16), (17) and (18) later).

Comparing the formulae given in Theorem 2(i) and in Theorem 1.1 of [11] shows that  $H_x^{I(n)}(s)/H_{x-n}^-(s-n) \rightarrow 1$  if  $x \notin I(n)$  and  $x - s = o(n)$ . The situation for the case  $x \in I(n)$  is different, as is exhibited in the next theorem. We extend  $h_x^{I(n)}(s)$  to the variables  $x \in I(n)$  by

$$h_x^{I(n)}(s) = \frac{n^2 - xs}{\pi(x - s)^2 \sqrt{(n_*^2 - x^2)(n_*^2 - s^2)}} \quad x, s \in I(n).$$

**Theorem 5** Suppose  $E[|S_1^{(1)}|^2 \log |S_1^{(1)}|] < \infty$ , where  $S_1^{(1)}$  denotes the first component of  $S_1$ . Let  $x, s \in I(n)$ . Then

(i) As  $(n - |s|) \wedge (n - |x|) \wedge |x - s| \rightarrow \infty$

$$H_x^{I(n)}(s) = \sigma^2 h_x^{I(n)}(s)(1 + o(1)).$$

(ii) Let  $0 \leq x < n$ . As  $(n - x)/(n - s) \rightarrow 0$

$$H_x^{I(n)}(s) = \frac{\sigma^2}{\pi} \cdot \frac{\nu(-n + x)\nu(-n - s)\sqrt{n}}{\sqrt{2}(x - s)^{3/2}}(1 + o(1)).$$

(ii') Let  $0 \leq s < n$ . As  $(n - s)/(n - x) \rightarrow 0$

$$H_x^{I(n)}(s) = \frac{\sigma^2}{\pi} \cdot \frac{\mu(-n + s)\mu(-n - x)\sqrt{n}}{\sqrt{2}(s - x)^{3/2}}(1 + o(1)).$$

Since in (ii)  $(x - s)/(n - s) \rightarrow 1$  and  $x/n \rightarrow 1$ , the formula of (ii) reduces to that of (i) if  $n - x, n + s \rightarrow \infty$  and similarly for (ii'). Such consideration also leads to the following corollary.

**Corollary 6** For  $x, s \in I(n)$ ,  $C^{-1}h_x^{I(n)}(s) \leq H_x^{I(n)}(s) \leq Ch_x^{I(n)}(s)$ . In particular if  $x$  stays in a bounded distance from  $-n$  and  $s$  does from  $n$ , then  $H_x^{I(n)}(s) \asymp 1/n$ .

REMARK. Under the same supposition as in Theorem 5 the formulae obtained above can be extended to the general starting positions  $x + iy$  as in [11] but with the resulting formula somewhat complicated (see 31).

For the symmetric simple random walk (i.e.,  $P_0[S_1 = x] = 1/4$  for  $x \in \{\pm 1, \pm i\}$ ) we can improve the estimate of  $H_x^\pm(s)$  in [11] and accordingly the estimate of  $H_x^{I(n)}(s)$  in Theorem 1.

**Proposition 7** Let  $S_n$  be the symmetric simple random walk. Then

$$H_x^-(s) = \frac{1}{\pi} \cdot \frac{1}{x - s} \sqrt{\frac{x \vee 1}{-s}} \times \left[ 1 + O\left(\frac{1}{-s}\right) + O\left(\frac{1}{x \vee 1}\right) \right] \quad (x \geq 0, s < 0) \quad (3)$$

and as  $n \rightarrow \infty$

$$H_x^{I(n)}(s) = h_x^{I(n)}(s) \left[ 1 + O\left(\frac{1}{(|x| - n_*) \wedge (n - |s|)}\right) \right] \quad (|x| \geq n, s \in I(n)). \quad (4)$$

Here the error estimates are uniform for integers  $x, s$  subject to the respective constraints indicated in parentheses.

Theorem 1 is proved in Section 2 by taking for granted certain several results whose proofs are given in Section 3. Proof of Proposition 7 is given at the end of Section 3. In Section 4 we make detailed estimation of  $H_x^{I(n)}(s)$  when  $x$  or  $s$  are near the edges of  $I(n)$  under the moment condition (2); in particular Theorem 2 is proved. Theorem 5 is proved in Section 5.

## 2 Proof of Theorem 1

Throughout this section we pick up and fix a (large) positive integer  $n$ , which we shall not designate in the notation introduced in this section even though it depends on  $n$ . Let  $\mathbf{1}(S)$  stand for the indicator of a statement  $S$ :  $\mathbf{1}(S) = 1$  or  $0$  according as  $S$  is true or not. Define for integers  $x \geq n$  and  $y > -n$ ,

$$Q(x, y) = \sum_{s=-\infty}^{-n} H_{x-n}^-(s-n) H_{s+n}^+(y+n),$$

$$Q_I(x, y) = Q(x, y) \mathbf{1}(-n < y < n),$$

$$K_I(x, y) = H_{x-n}^-(y-n) \mathbf{1}(-n < y < n),$$

and  $Q^0 = \mathbf{1}$  (the identity matrix),  $Q^1 = Q$  and inductively

$$Q^k(x, y) = \sum_{u=n}^{\infty} Q^{k-1}(x, u) Q(u, y) \quad (k = 1, 2, \dots), \quad (5)$$

and finally

$$\Lambda(x, y) = \sum_{k=1}^{\infty} Q^k(x, y) \mathbf{1}(y \geq n).$$

Then for  $x \geq n$ ,  $-n < s < n$ ,

$$\begin{aligned} H_x^{I(n)}(s) &= (1 + \Lambda)(Q_I + K_I)(x, s) \\ &= \sum_{y=n}^{\infty} [\mathbf{1}(y = x) + \Lambda(x, y)] [Q_I(y, s) + K_I(y, s)]. \end{aligned} \quad (6)$$

These are probabilities with self-evident meaning. We are to compare them with the corresponding ones, denoted by  $q, k_I, q_I$  and  $\lambda$ , for the standard two dimensional Brownian motion  $B(t)$ . In doing this it is recalled that the interval  $I(n)$  is defined to be  $(-n+1/2, n-1/2)$  instead of  $[-n+1, n-1]$ , which makes difference in the associated probabilities of the Brownian motion. Put  $L_{\pm} = \{t \in \mathbf{R} : \pm t > 0\}$  and  $\tau_{L_{\pm}} = \inf\{t > 0 : B(t) \in L_{\pm}\}$ , and define

$$h_x^{\pm}(s) = P_x^{BM}[B(\tau_{L_{\pm}}) \in ds]/ds \quad (x \in L_{\mp}, \pm s > 0),$$

where  $P_z^{BM}$  denotes the law of  $B(t)$  starting at  $z$ . Then for real  $x > n_*$ ,  $y > -n_*$ ,

$$q(x, y) = \int_{-\infty}^{-n_*} h_{x-n_*}^-(s-n_*) h_{s+n_*}^+(y+n_*) ds,$$

$$q_I(x, y) = q(x, y) \mathbf{1}(-n_* < y < n_*), \quad k_I(x, y) = h_{x-n_*}^-(y-n_*) \mathbf{1}(-n_* < y < n_*)$$

and  $q^k$  and  $\lambda$  are given in analogous ways; in particular  $q^1 = q$  and

$$\lambda(x, y) = \sum_{k=1}^{\infty} q^k(x, y) \mathbf{1}(y > n_*).$$

We know that

$$h_x^-(s) = \frac{\sqrt{x}}{\pi(x-s)} \cdot \frac{1}{\sqrt{-s}} \quad (x > 0, s < 0), \quad (7)$$

$$h_x^{I(n)}(s) = \frac{\sqrt{x^2 - n_*^2}}{\pi(x-s)} \cdot \frac{1}{\sqrt{n_*^2 - s^2}} \quad (x > n_*, -n_* < s < n_*).$$

The function  $Q$  is extended to that of reals by

$$Q(u, v) = Q(x, y) \quad \text{for} \quad (u, v) \in (x - \frac{1}{2}, x + \frac{1}{2}] \times (y - \frac{1}{2}, y + \frac{1}{2}].$$

With  $Q$  thus extended put

$$\eta = Q - q.$$

We shall prove the following relations (I) through (VI). The symbol  $f \asymp g$  means that  $C^{-1}g \leq f \leq Cg$ . Here and in what follows  $C$  denotes a positive constant which may depend on the law  $P_0[S_1 = \cdot]$  but is independent of any variables  $x, n, y, s$  contained therein explicitly or inexplicitly and may change from line to line. The products of two functions (of two variables) are understood to be that of integral operators in an analogous way to (5): eg.,  $\eta q(x, y) = \int_{n_*}^{\infty} \eta(x, u)q(u, y)du$ .

Let  $x > n_*, y > -n_*$  and  $-n_* < s < n_*$ ;  $x, y, s$  are real numbers in (I) through (III).

$$(I) \quad q(x, y) \asymp \frac{\sqrt{x - n_*}}{\sqrt{n_* + y}} \cdot \left| \frac{1}{x - y} \log \frac{x + 2n}{y + 2n} \right|.$$

The function  $t^{-1} \log(1+t)$  is understood to be continuously extended to  $t = 0$ . It is noticed that  $1/b < (b-a)^{-1} \log(b/a) < 1/a$  for  $0 < a < b$ , whence  $(x-y)^{-1} \log[(x+2n)/(y+2n)] \asymp x^{-1}$  if  $|x-y| \leq 3n$ ; in particular it yields the bound of  $q_I$  given in the next item where we also display the explicit form of  $k_I$  for convenience.

$$(I') \quad q_I(x, s) \asymp \frac{\sqrt{x - n_*}}{\sqrt{n_* + s}} \cdot \frac{1}{x} \left( 1 + \log \frac{x}{n_*} \right);$$

$$k_I(x, s) = \frac{1}{\pi} \frac{\sqrt{x - n_*}}{(x - s)\sqrt{n_* - s}}.$$

$$(II) \quad \frac{|\eta(x, y)|}{q(x, y)} \begin{cases} = o(1) & \text{as } (x - n) \wedge (n + y) \rightarrow \infty & \text{if } \delta = 0 \\ \leq \frac{C}{\sqrt{(x - n_*) \wedge (n_* + y)}} & & \text{if } \delta > \frac{1}{2}. \end{cases}$$

$$(III) \quad |\eta|(\mathbf{1} + \lambda)(q_I + k_I)(x, s) \begin{cases} = \left[ \frac{1}{\sqrt{n_*^2 - s^2}} \wedge h_x^{I(n)}(s) \right] \times o(1) & \text{as } n \rightarrow \infty & \text{if } \delta = 0 \\ \leq C \frac{1 + (\log x/n_*)^2}{\sqrt{x} \sqrt{n_*^2 - s^2}} & & \text{if } \delta > \frac{1}{2}. \end{cases}$$

Here  $\mathbf{1}$  in (III) stands for the identity operator and  $\delta$  in (II) and (III) for the constant in (2); the action of (integral) operators is understood analogously to (6).

For integers  $x \geq n, -n < s < n$ ,

$$(IV) \quad \sum_{y=n}^{\infty} \Lambda(x, y) \leq C \sqrt{\frac{x - n_*}{x}},$$

$$(V) \quad \sum_{y=n}^{\infty} \Lambda(x, y) \frac{1}{y-s} \leq C \frac{1}{n} \sqrt{\frac{x-n_*}{x}} \cdot \log \frac{3n}{n-s},$$

$$(VI) \quad \sum_{y=n}^{n+N} \Lambda(x, y) \leq C \sqrt{\frac{x-n_*}{x}} \cdot \frac{N}{n} \quad (N = 1, 2, \dots).$$

The proof of these results is postponed to the next section. In the rest of this section we prove Theorem 1 taking them for granted.

From the identity  $H^{I(n)} = (\mathbf{1} + \Lambda)(K_I + Q_I)$  and a similar one for  $h^{I(n)}$  it follows that

$$H^{I(n)} - h^{I(n)} = (\mathbf{1} + \Lambda)(K_I - k_I + Q_I - q_I) + (\Lambda - \lambda)(k_I + q_I).$$

Writing  $Q = \Lambda - \Lambda Q$  and  $q = \lambda - q\lambda$  one finds the identity  $\Lambda q - Q\lambda = \Lambda\eta\lambda$ , which yields

$$\Lambda - \lambda = \eta + \Lambda\eta + \eta\lambda + \Lambda\eta\lambda = (\mathbf{1} + \Lambda)\eta(\mathbf{1} + \lambda). \quad (8)$$

Let  $q_I + k_I$  act on the both sides from the right. Let  $x$  and  $s$  be integers such that  $x \geq n$  and  $-n < s < n$ . Using (III), (IV) and the simple inequality

$$x^{-\alpha} \log x/n \leq (e\alpha)^{-1} n^{-\alpha} \quad (\alpha > 0),$$

first observe that

$$\Lambda|\eta|(\mathbf{1} + \lambda)(q_I + k_I)(x, s) \leq \sqrt{\frac{x-n_*}{x}} \cdot \frac{\varepsilon(n)}{\sqrt{n^2 - s^2}},$$

where  $\varepsilon(t)$  is a function of (a single variable)  $t \geq 0$  such that as  $t \rightarrow \infty$ ,  $\varepsilon(t) = o(1)$  or  $O(1/\sqrt{t})$  according as  $\delta = 0$  or  $\delta > 1/2$  in (2), and then, further using (8) and (III), that

$$\begin{aligned} |\Lambda - \lambda|(q_I + k_I)(x, s) &\leq \left[ \varepsilon(n) \sqrt{x-n_*} + C \left( 1 + \log \frac{x}{n} \right)^2 \right] \frac{1}{\sqrt{x(n^2 - s^2)}} \\ &\leq C' \left( \varepsilon(n) + \frac{1}{\sqrt{x-n_*}} \right) h_x^{I(n)}(s). \end{aligned} \quad (9)$$

The last inequality in particular implies

$$\Lambda(k_I + q_I)(x, s) \leq C h_x^{I(n)}(s). \quad (10)$$

By the formula (16)

$$|k_I - K_I|(y, s) \leq k_I(y, s) [\varepsilon_1(y - n_*) + \varepsilon_2(n - s)], \quad (11)$$

where  $\varepsilon_1(t)$  and  $\varepsilon_2$  are functions of the same meaning as  $\varepsilon(t)$  in (9).

Let  $\delta > 1/2$  in (2). Combined with (V) and (10) the last bound shows that

$$\Lambda|k_I - K_I|(x, s) \leq C \left( \frac{1}{n} \sqrt{\frac{x-n_*}{x}} \log \frac{3n}{n-s} + h_x^{I(n)}(s) \right) \frac{1}{\sqrt{n-s}},$$

but we have  $n^{-1} \log[3n/(n-s)] \leq 1/\sqrt{n(n-s)}$  so that

$$\Lambda|k_I - K_I|(x, s) \leq C \left( \sqrt{\frac{x - n_*}{nx(x-s)}} + h_x^{I(n)}(s) \right) \frac{1}{\sqrt{n-s}} \leq C' h_x^{I(n)}(s) \frac{1}{\sqrt{n-s}}. \quad (12)$$

On the other hand by (II) and (I') we have

$$|q_I - Q_I|(y, s) \leq C \left( \frac{1}{\sqrt{y-n}} + \frac{1}{\sqrt{n+s}} \right) \cdot q_I(y, s) \leq C \frac{1 + \log y/n}{\sqrt{y}} \cdot \frac{1}{n+s} \quad (13)$$

and  $\Lambda|q_I - Q_I| \leq C \sqrt{x - n_*}/\sqrt{xn}(n+s) \leq C' h_x^{I(n)}(s)/\sqrt{n+s}$ , which combined with (12) gives

$$(\mathbf{1} + \Lambda) \left( |q_I - Q_I| + |k_I - K_I| \right)(x, s) \leq C h_x^{I(n)}(s) \left( \frac{1}{\sqrt{n-s}} + \frac{1}{\sqrt{n+s}} \right). \quad (14)$$

The bounds (9), (11), (13) and (14) together yield the formula of Theorem 1 in the case  $\delta > 1/2$ .

### 3 Proofs of (I) through (VI)

*Proof of (I).* Let  $x \geq n_*$  and  $y > -n_*$ . It follows from (7) that

$$\begin{aligned} q(x, y) &= \frac{1}{\pi^2} \int_{-\infty}^{-n_*} \frac{\sqrt{x - n_*}}{(x - u)\sqrt{-u + n_*}} \frac{\sqrt{-u - n_*}}{(y - u)\sqrt{n_* + y}} du \\ &= \frac{1}{\pi^2} \sqrt{\frac{x - n_*}{y + n_*}} J_n(x + n_*, y + n_*), \end{aligned} \quad (15)$$

where

$$J_n(a, b) = \int_0^\infty \frac{\sqrt{t} dt}{\sqrt{t + 2n_*}(t + a)(t + b)} \quad (a = y + n_* > 0, \ b = x + n_* > 2n_*).$$

If  $n_* \leq a < b$ , then

$$\begin{aligned} J_n(a, b) &= \frac{1}{a} \int_0^\infty \frac{\sqrt{t} dt}{\sqrt{t + 2n_*/a}(t + 1)(t + b/a)} \\ &\asymp \frac{1}{b} + \frac{1}{a} \int_1^\infty \frac{dt}{(t + 1)(t + b/a)} = \frac{1}{b} + \frac{1}{b - a} \log \frac{a + b}{2a} \asymp \frac{1}{b - a} \log \frac{b}{a}. \end{aligned}$$

This shows (I) in the case  $y \geq n_*$ .

In the case  $-n_* < y < n_*$ , a similar computation gives

$$J_n(a, b) = \frac{1}{2n_*} \int_0^\infty \frac{\sqrt{t} dt}{\sqrt{t + 1}(t + a/2n_*)(t + b/2n_*)} \asymp \frac{1}{b} + \frac{1}{b - a} \log \frac{2n_* + b}{2n_* + a}.$$

Thus (I) has been proved.

*Proof of (II).* First suppose that  $\delta > 1/2$  in (2). Then it is shown in [11] (Theorem 3) that there exists a constant  $C$  such that for  $x \geq n$  and  $s < n$ ,

$$\left| H_{x-n}^-(s-n) - \frac{1}{\pi} \cdot \frac{1}{x-s} \sqrt{\frac{x-n_*}{n-s}} \right| \leq C h_{x-n_*}^-(s-n_*) \left( \frac{1}{\sqrt{n_*-s}} + \frac{1}{\sqrt{x-n_*}} \right). \quad (16)$$

In making application of this and its obvious analogue for  $H^+$  there arise four terms to be estimated for computation of the difference  $Q - q = H^- H^+ - h^- h^+ = (H^- - h^-) H^+ + h^- (H^+ - h^+)$  (the right side of (16) is counted two terms), which are equal to those obtained by inserting the factors

$$\frac{1}{\sqrt{x-n_*}} + \frac{1}{\sqrt{-u+n_*}} \quad \text{and} \quad \frac{1}{\sqrt{n_*+y}} + \frac{1}{\sqrt{-u-n_*}}$$

under the integral symbol of the integral of (15). Among them only two terms require computation, which we are to show to be not larger than the sum of the other two. To this end, we make the same change of variables that led to the second equality of (15) and find that it suffices in view of (I) to verify the following inequalities

$$\int_0^\infty \frac{\sqrt{t} dt}{(t+2n_*)(t+a)(t+b)} \leq \int_0^\infty \frac{dt}{\sqrt{t+2n_*}(t+a)(t+b)} \leq \frac{\pi}{(a \vee b) \sqrt{a \wedge b}}$$

( $a = y + n_* > 0$ ,  $b = x + n_*$  as before). The first one is trivial. The second one is verified by dominating the integral in the middle by  $(a \vee b)^{-1} \int_0^\infty [\sqrt{t}(t+a \wedge b)]^{-1} dt$ . This completes the proof in the case  $\delta > 1/2$ .

The case  $\delta = 0$  is similarly dealt with based on the corresponding result on  $H_x^-(s)$  (Theorem 1 of [11]).

*Proof of (III).* Consider the case  $\delta > 1/2$ . Set

$$A(y) = \frac{1}{\sqrt{x-n_*}} q(x, y) h_y^{I(n)}(s) \sqrt{n_*^2 - s^2},$$

$$B(y) = q(x, y) \frac{1}{\sqrt{y+n_*}} h_y^{I(n)}(s) \sqrt{n_*^2 - s^2}$$

and

$$I_A = \int_{(x-2n) \vee n}^\infty A(y) dy \quad \text{and} \quad I_B = \int_{n_*}^{(x-2n) \vee n} B(y) dy.$$

Notice that  $1/\sqrt{x-n_*} \leq 1/\sqrt{y+n_*}$  if and only if  $y \leq x-2n_*$  and that according to (II)

$$|\eta| h^{I(n)}(x, s) \leq C(I_A + I_B) / \sqrt{n_*^2 - s^2}$$

By (I)

$$A(y) \asymp \frac{\sqrt{y-n_*}}{y-s} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n}.$$

Simple computation shows that

$$\int_{(x-2n) \vee n}^{(x-2n) \vee (2n)} A(y) dy = O(1/\sqrt{x})$$



and

$$\int_{(x-2n) \vee (2n)}^{\infty} A(y) dy \leq C \int_{(x-2n) \vee (2n)}^{\infty} \frac{1}{y^{3/2}} \left(1 + \log \frac{y}{x}\right) dy = O(1/\sqrt{x}).$$

Thus  $I_A = O(1/\sqrt{x})$  (uniformly in  $s < n_*$ ).

For  $B$  we have

$$B(y) \asymp \frac{\sqrt{x-n_*}}{\sqrt{y+n_*}} \frac{\sqrt{y-n_*}}{y-s} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n}.$$

Let  $x \geq 3n$ . It is easy to see that  $\int_n^{2n} B(y) dy = \text{const } (1/\sqrt{x}) \log(x/n_*)$ , while

$$\begin{aligned} \int_{2n}^x B(y) dy &\leq C\sqrt{x} \int_{2n}^x \frac{1}{y} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n} dy \\ &\leq C \frac{2\sqrt{x}}{x+2n} \int_{4n/(x+2n)}^1 \frac{1}{u(1-u)} \log \frac{1}{u} du \end{aligned}$$

and we can dominate the last member by a constant multiple of  $(1/\sqrt{x})(\log x/n)^2$  owing to the equality  $\int_a^1 u^{-1} \log(1/u) du = \frac{1}{2} |\log a|^2$  ( $0 < a < 1$ ). This verifies (III) when  $\delta > 1/2$ . For the case  $\delta = 0$  the same argument as above gives the upper bound  $o(1)/\sqrt{n_*^2 - s^2}$ ; the identity  $qh^I = h^I - (k_I + q_I)$  ( $\leq h^I$ ) gives the other bound  $h^I \times o(1)$ . The proof of (III) is complete.

*Proof of (IV).* Put  $p_n = \sup_{x,y \geq n} Q(x, y)$ . Then  $p_n = O(1/\sqrt{n})$  and

$$\sum_{y \geq n} Q^k(x, y) \leq p_n^{k-1} \sum_{y \geq n} Q(x, y) \leq p_n^{k-1} \sum_{y \geq -n} H_{x-n}^-(y) \leq p_n^{k-1} C \sqrt{\frac{x-n_*}{x}},$$

hence  $\sum_{y=n}^{\infty} \Lambda(x, y) \leq \sum_k \sum_{y \geq n} Q^k(x, y) \leq C' \sqrt{(x-n_*)/x}$ .

**Lemma 8** *Uniformly for integers  $x \geq n$ ,  $-n < s < n$ ,*

$$\sum_{y=n}^{\infty} Q(x, y) \frac{1}{y-s} \asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right) \log \frac{3n}{n-s}.$$

*Proof.* By employing (I) and the bound  $(x-y)^{-1} \log \frac{x+2n}{y+2n} \asymp x^{-1} [1 + \log(x/y)]$  valid for  $n \leq y \leq x+n$  one sees that

$$\begin{aligned} \sum_{y=n}^{2n} Q(x, y) \frac{1}{y-s} &\asymp \int_n^{2n} \frac{\sqrt{x-n_*}}{(y-s)\sqrt{n+y}} \cdot \frac{1}{x-y} \log \frac{x+2n}{y+2n} dy \\ &\asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right) \log \frac{2n-s}{n-s} \end{aligned}$$

as well as

$$\sum_{y=2n+1}^{\infty} Q(x, y) \frac{1}{y-s} \leq C \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right)$$

(divide the summation according as  $y$  is larger or smaller than  $x \vee (2n)$  and consider the cases  $x \leq 2n$  and  $x > 2n$  separately). These together yield the estimate of the lemma.  $\square$

*Proof of (V).* Dominating by  $n^{-1} \log[3n/(n-s)]$  the right-hand side of the asymptotic formula of Lemma 8 and employing (IV), we have

$$\sum_{y=n}^{\infty} \Lambda(x, y) \frac{1}{y-s} = \sum_{y=n}^{\infty} (Q + \Lambda Q)(x, y) \frac{1}{y-s} \leq C \sqrt{\frac{x-n_*}{x}} \frac{\log[3n/(n-s)]}{n}.$$

Thus (V) is proved.

*Proof of (VI).* As in the proof of Lemma 8 we have  $\sum_{y=n}^{n+N} Q(x, y) \leq C \sqrt{(x-n_*)/x} N/n$ ; the estimate of (VI) is then follows from (IV) as in the preceding proof.

*Proof of Proposition 7.* Both formulae (3) and (4) of Proposition 7 are proved in a similar way as Theorem 1 and their proofs are given but with details omitted. We first give an outline of deduction of (4) from (3). For the symmetric simple random walk the right-hand side of (16) can be replaced by  $Ch_x^-(s)[(-s)^{-1} + (x \vee 1)^{-1}]$  (cf. [11]) and accordingly we deduce that

$$(|k_I - K_I| + |q_I - Q_I|)(y, s) \leq \frac{Ck_I(y, s)}{[(y-n_*) \wedge (n-s) \wedge (n+s)]},$$

$$|\eta|(x, y) \leq \frac{Cq(x, y)}{(x-n_*) \wedge (n_*+y)},$$

$$\sum_{y=n}^{\infty} \frac{Q(x, y)}{(y-s)\sqrt{y-n_*}} \asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \left(1 + \log \frac{x}{n}\right) \frac{1}{\sqrt{n-s}},$$

$$\sum_{y=n}^{\infty} \frac{\Lambda(x, y)}{(y-s)\sqrt{y-n_*}} \leq \frac{C\sqrt{x-n_*}}{n\sqrt{x}\sqrt{n-s}}$$

and

$$\sum_{y=n}^{\infty} \frac{\Lambda(x, y)k_I(y, s)}{y-n_*} \leq \frac{C\sqrt{x-n_*}}{\sqrt{x}n(n-s)},$$

and with these bounds we can proceed as above to obtain (4).

*Proof of (3):* The proof is based on an expansion of the potential function (cf. [2], [10], [6]) from which an application of the reflection principle immediately yields

$$H_{im}(s) = \frac{|m|}{\pi(|s|^2 + m^2)} + O\left(\frac{1}{|s|^3 + |m|^3}\right) \quad (m \neq 0),$$

where  $H_{im}(s)$  stands for the probability that the first visit to the real axis of the simple random walk starting at  $im \in i\mathbf{Z}$  takes place at  $s \in \mathbf{Z}$ . We proceed as in Section 2. Bearing symmetry of the walk in mind, this time we define for  $y \in \mathbf{Z}$  and  $x \geq 0$ ,

$$Q(x, y) = \sum_{m=-\infty}^{\infty} H_{ix}(m)H_{im}(y) \quad \text{if } x > 0 \quad \text{and} \quad = H_0(y) \quad \text{if } x = 0$$

and inductively  $Q^k(x, y) = \sum_{u=0}^{\infty} Q^{k-1}(x, u)Q(u, y)$  ( $k = 1, 2, \dots$ ). We have the corresponding quantities  $h, h^-, q$  and  $q^k$  for the standard Brownian motion. Then

$$H_x^-(s) = \Lambda(x, s) := \sum_{k=1}^{\infty} Q^k(x, s), \quad h_x^-(s) = \lambda(x, s) := \sum_{k=1}^{\infty} q^k(x, s) \quad (s < 0, \ x \geq 0).$$

We know that  $C^{-1}\lambda \leq \Lambda \leq C\lambda$  for some constant  $C > 0$  (cf. [11]). We suitably extending  $Q$  to the real variables and put  $\eta = Q - q$  as before. An elementary computation then gives in turn

$$\begin{aligned} \eta(x, y) &= O(|xy^2|_+^{-1} \wedge |x^2y|_+^{-1}), \\ |\eta|\lambda(x, y) &= O(|xy|_+^{-3/2} \wedge |x^2y|_+^{-1}), \quad \Lambda|\eta|(x, y) = O(|x^{1/2}y^2|_+^{-1}) \end{aligned}$$

and

$$\Lambda|\eta|\lambda(x, s) \leq \frac{C}{|x^{1/2}s^{3/2}|_+} \asymp \frac{1}{x-s} \sqrt{\frac{x \vee 1}{-s}} \times \left[ \frac{1}{-s} + \frac{1}{x \vee 1} \right] \quad (s \leq -1, \ x \geq 0),$$

where  $|a|_+ = |a| \vee 1$ . Thus (3) follows in view of the identity  $\Lambda - \lambda = (1 + \Lambda)\eta(1 + \lambda)$ .

The proof of Proposition 7 is complete.

## 4 Estimation of $H^{I(n)}$ near the edges

We continue the arguments of the preceding section to estimate  $H^{I(n)}$  mainly in the case when  $\delta > 1/2$  and either  $n - s$  or  $n + s$  is small in comparison with  $x - n$ . The case when  $\delta = 0$  or  $x - n$  is not large can be similarly dealt with and is only briefly discussed at the end of this section.

Theorems 9 and 10 given below are based on the following result from [11]: if  $\delta > 1/2$  in (2), then for  $x \geq s > 0$ ,

$$H_x^-(s) = \frac{\sqrt{x}}{\pi(x+s)} \mu^-(s) + O\left(\frac{1}{x}\right), \quad (17)$$

$$H_{-x}^+(s) = \frac{\sqrt{x}}{\pi(x+s)} \nu^-(s) + O\left(\frac{1}{x}\right), \quad (18)$$

where  $\mu^-(s) = \mu(-s)$  and  $\nu^-(s) = \nu(-s)$ . The following theorem concerns particularly to the case when  $(x - n)/(n - s) \rightarrow \infty$  so that  $h_x^{I(n)}(s) \gg (x - s)^{-1}$ .

**Theorem 9** *If  $\delta > 1/2$  in (2), then uniformly for integers  $n > 1, 0 \leq s < n$  and  $x \geq n$ ,*

$$H_x^{I(n)}(s) = \sqrt{n_* - s} \mu^-(n - s) h_x^{I(n)}(s) + O\left(\frac{\log n}{n} + \frac{1}{x - s}\right).$$

*Proof.* Make decomposition  $H_x^{I(n)} = K_I + Q_I + \Lambda(K_I + Q_I)$  and infer from (17) that

$$K_I = \sqrt{n_* - s} \mu^-(n - s) k_I + O(1/(x - s)), \quad (19)$$

$$\Lambda K_I(x, s) = \sum_{y=n}^{\infty} \Lambda(x, y) \left[ \sqrt{n_* - s} \mu^-(n - s) k_I(y, s) + O\left(\frac{1}{y - s}\right) \right].$$

In view of (I) we have  $\sup_{s \geq 0, x > n_*} q_I(x, s) \leq C/n$ , which in particular shows that

$$\lambda q_I(x, s) = O(1/n) \quad \text{uniformly for } 0 \leq s < n_*, x > n_*.$$

Thus, on employing (I'), for  $s > 0$ ,

$$q_I + \lambda q_I = O(1/n) \quad \text{and} \quad Q_I + \Lambda Q_I \leq C(q_I + \lambda q_I) = O(1/n). \quad (20)$$

By (V) of the preceding section we have

$$\sum_{y=n}^{\infty} \Lambda(x, y) \frac{1}{y-s} \leq C \sqrt{\frac{x-n_*}{x}} \frac{\log n}{n}, \quad (21)$$

so that

$$\Lambda K_I(x, s) = \sqrt{n_* - s} \mu^-(n-s) \Lambda k_I(x, s) + O(n^{-1} \log n).$$

Here the factor  $\sqrt{(x-n_*)/x}$  on the right side of (21) is replaced by 1: the loss of accuracy to the estimate of  $H_x^{I(n)}$  caused by this replacement is small in comparison with the error term  $O(1/(x-s))$  in (19). By (9)  $\Lambda k_I = \lambda k_I + (\Lambda - \lambda)k_I = \lambda k_I + O(1/n)$ ; hence

$$\Lambda K_I = \sqrt{n_* - s} \mu^-(n-s) \lambda k_I + O(n^{-1} \log n),$$

which together with (19), (20) yields the assertion of the theorem.  $\square$

**Theorem 10** *If  $\delta > 1/2$  in (2), then uniformly for integers  $n > 1$ ,  $-n < s < 0$  and  $x \geq n$ ,*

$$H_x^{I(n)}(s) = \sqrt{n_* + s} \nu^-(n+s) h_x^{I(n)}(s) \left[ 1 + O\left(\sqrt{\frac{s+n_*}{n}} \cdot \log n\right) + O\left(\sqrt{\frac{x}{n(x-n_*)}}\right) \right].$$

*Proof.* Make decomposition

$$\begin{aligned} H_x^{I(n)}(s) &= K_I(x, s) + \sum_{y=-\infty}^{-n} H_{x-n}^-(y-n) H_y^{I(n)}(s) \\ &= K_I(x, s) + \sum_{y=-\infty}^{-n} (H_{x-n}^- - h_{x-n}^-)(y-n) H_y^{I(n)}(s) \\ &\quad + \sum_{y=-\infty}^{-n} h_{x-n}^-(y-n) H_y^{I(n)}(s). \end{aligned} \quad (22)$$

For evaluation of the second sum of the last line we substitute the estimate of Theorem 9 (with  $S_n$  replaced by  $-S_n$ , hence  $\sqrt{n_* - s} \mu^-(n-s)$  by  $\sqrt{n_* + s} \nu^-(n+s)$ ) for  $H_x^{I(n)}(s)$ , use the expression of  $h_x^{I(n)}(s)$  analogous to the first expression in (22) of  $H_x^{I(n)}(s)$  and observe that

$$\sup_{-n < s \leq 0} k_I(x, s) \asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}}; \quad \int_{-\infty}^{-n} h_{x-n}^-(y-n) dy \asymp \frac{\sqrt{x-n_*}}{\sqrt{x}};$$

$$\text{and } \int_{-\infty}^{-n} h_{x-n}^-(y-n)(s-y)^{-1} dy \asymp \frac{\sqrt{x-n_*}}{x\sqrt{n}} \cdot \log \frac{n}{s+n}$$

to obtain

$$\sum_{y=-\infty}^{-n} h_{x-n}^-(y-n) H_y^{I(n)}(s) = \sqrt{n_*+s} \nu^-(n+s) h_x^{I(n)}(s) + O\left(\frac{\sqrt{x-n_*}}{\sqrt{x}n} \cdot \log n\right). \quad (23)$$

For evaluation of the first sum apply (16) to have  $|H_{x-n}^-(y') - h_{x-n}^-(y')| \leq C((x-n_*)^{-1/2} + n^{-1/2})h_{x-n}^-(y')$  for  $y' \leq -2n$ ; also use the bound  $H_x^{I(n)}(s) \leq C h_x^{I(n)}(s)$  that follows from Theorem 1. These bounds as well as  $\int_{-\infty}^{-n_*} h_{x-n}^-(y-n) h_y^{I(n)}(s) dy \leq h_x^{I(n)}(s)$  yield

$$\begin{aligned} \sum_{y=-\infty}^{-n} |(H_{x-n}^- - h_{x-n}^-)(y-n)| H_y^{I(n)}(s) &\leq C \left( \frac{1}{\sqrt{x-n_*}} + \frac{1}{\sqrt{n}} \right) h_x^{I(n)}(s) \\ &\asymp \frac{\sqrt{x}}{\sqrt{(x-n_*)n}} h_x^{I(n)}(s). \end{aligned}$$

Combined with (22) and (23) this completes the proof of the theorem.  $\square$

As being mentioned at the beginning of this section our estimation of  $H^{I(n)}$  made above is appropriate if  $x-n$  is large in comparison with  $n \pm s$ . When  $x-n$  is not large, it is better to replace  $h_{x-n}^-(y-n)$  by

$$\frac{\sigma^2 \nu(x-n)}{2\sqrt{x-n_*}} h_{x-n_*}^-(y-n)$$

in (22); also make use of the corresponding estimate of  $H_x^\pm$  in [11].

*Proof of Theorem 2.* The proof is based on an estimate of  $H_x^-(s)$  verified in [11] (Theorem 1 of it). The case when  $(x-n) \wedge (n-s) \wedge (n+s) \rightarrow \infty$  the assertion is included in Theorem 1. The other case is dealt with as in the proofs of Theorems 9 and 10 by employing (VI). The computations to be carried out this time are much simpler in either case.

## 5 Proof of Theorem 5

We always have the relation

$$H_x^{I(n)}(s) = H_x(s) + \sum_{|x_1| \geq n} H_x(x_1) H_{x_1}^{I(n)}(s) \quad s \in I(n), \quad (24)$$

where  $H_x(s)$ ,  $x, s \in \mathbf{Z}$ , denote the probability that  $s$  is the site where the real axis is hit for the first time after time 0 by the walk  $S_n$  started at  $x$ . Suppose  $E[|S_1^{(1)}|^2 \log |S^{(1)}|] < \infty$ . Then

$$H_x(s) = H_0(s-x) = \frac{\sigma^2}{\pi(s-x)^2} (1 + o(1)) \quad \text{as } |s-x| \rightarrow \infty \quad (25)$$

(cf. [12]). Theorem 5 is derived by combining Theorem 2 and (25). Before proceeding into the proof we mention several points to be recognized. For the estimate of  $H_x^{I(n)}(s)$ ,  $s, x \in$

$I(n)$  we may suppose that  $x \geq 0$  for obvious reason and then,  $s < x$ , by considering the time-reversed walk. Also (ii') is a dual statement of (ii). Moreover there are some possibility of improving the estimates in certain cases that we do not take up in this paper, and for that some details given below may be helpful .

As noticed above we may suppose

$$s < x \quad \text{and} \quad x \geq 0. \quad (26)$$

Then, according to Theorem 2, for  $x_1 \geq n$ ,

$$H_{x_1}^{I(n)}(s) = \left[ \frac{\nu(-n-s)}{\sqrt{n-s}} \right] \frac{\sigma^2 \nu(x_1-n) \sqrt{n+x_1}}{2\pi|x_1-s|} (1+o(1)) \quad (27)$$

$$= \left[ \sqrt{n+s} \nu(-n-s) \right] h_{x_1}^{I(n)}(s) \frac{\sigma^2 \nu(x_1-n)}{2\sqrt{x_1-n}} (1+o(1)). \quad (28)$$

*Proof of (ii).* The function  $\nu$  satisfies  $\sum_{k=0}^{\infty} H_0(-j+k) \nu(k) = \nu(j)$  for all  $j \in \mathbf{Z}$  (see the definition of  $\nu$  in [11]), so that

$$\sum_{x_1=n}^{\infty} H_x(x_1) \nu(x_1-n) = \nu(-n+x). \quad (29)$$

We claim that as  $(n-x)/(n-s) \rightarrow 0$ ,

$$J_{n,s,x} := \sum_{x_1 \geq n} H_x(x_1) H_{x_1}^{I(n)}(s) = \left[ \frac{\nu(-n-s)}{\sqrt{n-s}} \right] \cdot \frac{\sigma^2 \nu(-n+x) \sqrt{2n}}{2\pi(n-s)} (1+o(1)). \quad (30)$$

Put  $\xi = n-x$  and observe that the summation in (29) may be restricted to  $x_1 \leq n+K\xi$  by choosing  $K$  large enough. Then, on looking at (27), the claim (30) follows if we show that for each  $\varepsilon > 0$  we can find  $K > 1$  such that

$$\sum_{x_1 \geq n+K\xi} H_x(x_1) H_{x_1}^{I(n)}(s) \leq \varepsilon J_{n,s,x}.$$

However, by simple consideration this reduces to

$$\int_{K\xi}^{\infty} \frac{\sqrt{2n+u}}{(u+\xi)^2(u+n-s)\sqrt{u}} du \leq \frac{\varepsilon \sqrt{n}}{(n-s)\sqrt{\xi}},$$

which is certainly true if  $K$  is large enough. Thus the claim is verified. One can easily check that  $\sum_{x_1 \leq -n} H_x(x_1) H_{x_1}^{I(n)}(s) = o(J_{n,s,x})$ . Finally notice that  $(n-s)/(x-s) \rightarrow 1$  and hence the right side of (30) may be identical to that of the required formula. The proof of (ii) is complete.  $\square$

*Proof of (i).* We may suppose that  $(n-x)/(n-s)$  is bounded away from zero, the case  $(n-x)/(n-s) \rightarrow 0$  being included in (ii) that is proved above, Under this condition we see

$$\sum_{n \leq |x_1| \leq n+K} H_x(x_1) H_{x_1}^{I(n)}(s) = o\left(\frac{1}{(s-x)^2}\right)$$

(indeed this is valid if  $(n-x)^4/(n-s) \rightarrow \infty$ ; the contribution of  $-n-K \leq x_1 \leq -n$  is easy to estimate), namely the sum above is negligible if compared with  $H_x(s)$ . Hence one can replace the ratio appearing last in (28) by 1. Also  $\nu(-n-s)$  may be replaced by  $1/\sqrt{n+s}$  and, substituting the resulting expression into (24) and applying Lemma 11 of Appendix, we conclude the formula of (i).  $\square$

## 6 Appendix

Let  $D$  be the complement of the line segment with edges at  $\pm 1$ :

$$D = \mathbf{C} \setminus \{s : -1 \leq s \leq 1\}.$$

The function  $z = \frac{1}{2}(w + w^{-1})$  univalently maps the exterior of the unit circle onto  $D$ . Denote by  $f(z)$  its inverse map, which may be represented by  $f(z) = z + \sqrt{z^2 - 1}$  (with the standard choice of the branch of the square root, so that  $f(\pm s) = s \pm \sqrt{s^2 - 1}$  for  $s > 1$  and  $f(s \pm i0) = s \pm i\sqrt{1-s^2}$  for  $-1 < s < 1$ ). As  $w = f(z)$  moves on a circle centered at the origin counter-clockwise and starting at a point  $R > 1$ ,  $z$  describes the ellipse  $[2x/(R+R^{-1})]^2 + [2y/(R-R^{-1})]^2 = 1$  (which surrounds the segment  $-1 \leq s \leq 1$  and shrinks to it as  $R \downarrow 1$ ) rotating also counter-clockwise and starting at the point  $f(R) = \frac{1}{2}(R+R^{-1}) \in (1, R)$ . Since the Poisson kernel for the exterior of the unit circle is given by  $K(Re^{i\theta}, \theta') = (2\pi)^{-1}(R^2 - 1)/(R^2 - 2R\cos(\theta - \theta') + 1)$ , if

$$R(z) = |f(z)|, \quad \theta(z) = \arg f(z),$$

then for  $-1 \leq s \leq 1$ ,  $\theta(s \pm i0) = \pm \arccos s \in (-\pi, \pi)$ , so that  $|d\theta(s \pm i0)| = ds/\sqrt{1-s^2}$ , and

$$h_D(z, s \pm i0) = \frac{1}{2\pi} \cdot \frac{R^2(z) - 1}{R^2(z) - 2R(z)\cos[\theta(z) - \theta(s \pm i0)] + 1} \cdot \frac{1}{\sqrt{1-s^2}}, \quad (31)$$

which for  $z = x \in \mathbf{R} \setminus [-1, 1]$  reduces to

$$h_D(x, s) := 2h_D(x, s + i0) = \frac{\sqrt{x^2 - 1}}{\pi|x-s|} \cdot \frac{1}{\sqrt{1-s^2}}.$$

Let  $Q = (\sigma_{ij})$  be a  $2 \times 2$  matrix that is symmetric and positive definite and  $\tilde{h}_D(z, s \pm i0)$  the corresponding hitting density for the process  $X_t = Q^{1/2}B_t$ . Then

$$\tilde{h}_D(z, s \pm i0) = h_D(\tilde{z}, s \pm i0), \quad \tilde{z} = (x - \omega y) + i\lambda y, \quad (32)$$

where  $\omega = \sigma_{12}/\sigma_{22}$  and  $\lambda = (\det Q)^{1/2}/\sigma_{22}$ . In Section 5 we have used the following lemma.

**Lemma 11** For  $x, s \in I(n)$ ,

$$\frac{1}{(s-x)^2} + \int_{|\xi| \geq n_*} \frac{1}{(\xi-x)^2} h_\xi^{I(n)}(s) d\xi = \frac{n_*^2 - xs}{(s-x)^2 \sqrt{(n_*^2 - x^2)(n_*^2 - s^2)}}. \quad (33)$$

*Proof.* By the scaling property we may suppose that  $n_* = 1$ . Let  $h_y(x)$  denote the Poisson kernel on the upper half plane:  $h_y(x) = y/\pi(y^2 + x^2)$ . Then

$$h_D(z, s) := h_D(z, s \pm i0) + h_D(z, s \pm i0) = h_y(s - x) + \int_{|\xi|>1} h_y(\xi - x) h_D(\xi, s) d\xi,$$

which shows that  $\lim_{y \downarrow 0} \pi y^{-1} h_D(x + iy, s)$  equals L.H.S. of (33). The lemma therefore follows if we verify

$$\lim_{y \downarrow 0} \frac{\pi h_D(x + iy, s)}{y} = \text{R.H.S. of (33)}. \quad (34)$$

If  $w = -(1 - x^2 + y^2) + i2xy$  and  $\phi = \pi - \arg w \in (-\pi/2, \pi/2)$ , then

$$|f(z)|^2 = (x + |w|^{1/2} \sin \phi)^2 + (y + |w|^{1/2} \cos \phi)^2$$

and we see that  $y^{-1}(|f(z)|^2 - 1) \rightarrow 2/\sqrt{1 - x^2}$ . In view of (31) this shows that

$$\lim_{y \downarrow 0} \frac{\pi h_D(x + iy, s \pm i0)}{y} = \frac{1}{2(1 - \cos(\theta_x \mp \theta_s)) \sqrt{(1 - x^2)(1 - s^2)}}$$

where for  $-1 < t < 1$ ,  $\cos \theta_t = t$  with  $\theta_t \in (0, \pi)$ . Now (34) follows from the identity

$$\frac{1}{1 - \cos(\theta_x - \theta_s)} + \frac{1}{1 - \cos(\theta_x + \theta_s)} = \frac{2 - 2 \cos \theta_x \cos \theta_s}{(\cos \theta_x - \cos \theta_s)^2} = \frac{2(1 - xs)}{(x - s)^2}.$$

□

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